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Kramers Time in Weakly Nonpotential Systems[1]

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We consider a bistable Fokker–Planck system with a known stationary distribution and a small nonpotential part in the drift force. We perform a perturbation calculation of its Kramers time, τ_K, and compare it with the corresponding time, $\tau_K^{(0)}$, for the potential system which has the same stationary distribution. We show that $\tau_K/\tau_K^{(0)}$ depends only on the properties of the drift force close to the "saddle-point."

KEY WORDS: Nonpotential drift force; nonlinear Fokker–Planck equation; bistable stochastic systems; Kramers time.

1. INTRODUCTION

Fluctuation and relaxation effects have been extensively studied in stochastic systems (in particular those described by a Fokker–Planck equation) driven by nonlinear deterministic forces which derive from a "generalized potential." These are the systems which obey a "generalized free energy" minimization principle.

However, up to now, very little is known about the corresponding properties of nonpotential systems. Graham[(1)] has studied their stationary distribution P_{st}, and has shown the following: (1) If the stationary distribution is invariant with respect to time reversal, one can calculate P_{st} explicitly by means of a single quadrature. This case, in Graham's terminology, is that of "manifest detailed balance." (2) If such is not the case, there always exists a "general time-reversal" transformation with respect to which

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any Fokker–Planck equation has an invariant stationary distribution, provided that P_{st} exists.

These properties are connected with the fact that, given the Fokker–Planck dynamics, a given P_{st} is not associated with a unique drift force, but with an infinite class of them, one of which only derives from a potential.

Indeed, the normalized stationary distribution

$$P_{st} = N\exp(-U/\theta) \tag{1}$$

satisfies the Fokker–Planck equation

$$\frac{\partial P}{\partial t} = \nabla\cdot(-\mathbf{K}P + \theta\nabla P) \tag{2}$$

(we assume that the diffusion tensor $\bar{\bar{\theta}}$ is constant, diagonal, and isotropic) for all deterministic forces $\mathbf{K}$ such that

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_1 \tag{3}$$

with

$$\mathbf{K}_0 = -\nabla U \tag{4a}$$

$$\nabla\cdot\mathbf{K}_1 - (1/\theta)\mathbf{K}_1\cdot\nabla U = 0 \tag{4b}$$

On the other hand, to our knowledge, only one general result is known about the dynamics of the relaxation of nonpotential systems towards their stationary state, which is due to Risken.[2] He has shown that, among all the systems corresponding to a same stationary distribution P_{st}, the longest relaxation time τ is maximum for the potential system.

This result is of particular interest for bistable systems with two (or more) stable fixed points: in this case, τ is Kramers time τ_K, which characterizes the dynamics of population exchanges between the two basins of attraction. When $\Delta U = U_{max} - U_{min} \gg \theta$, τ_K is considerably larger than any other characteristic time of the system, and thus describes the final stage of the relaxation process.

It would obviously be interesting to go one step further and calculate τ_K explicitly, at least in simple cases. In this paper, we perform such a calculation for weakly nonpotential systems, for which the nonpotential part of the drift force, $\mathbf{K}_1$, can be treated as a perturbation.

This question was also recently addressed by Gardiner.[3] He has worked out a method which generalizes to a nonpotential bistable system with a known stationary distribution the Landauer–Swanson[4] expression of τ_K for a multidimensional potential system. His approach involves a nontrivial assumption.

We show at the end of Section 2 that the perturbation approach justifies his method, and discuss its physical meaning.

2. PERTURBATIVE CALCULATION OF KRAMERS TIME

We consider a bistable system with a drift force **K** and a known stationary distribution [Eq. (1)]. We assume that the minima of U (the two peaks of P_{st}) are located at $\mathbf{r} = \mathbf{a}, \mathbf{b}$, and that these minima are connected by a saddle point located at $\mathbf{r} = 0$.

K can be decomposed, according to Eqs. (4), into a potential and a nonpotential part, $\mathbf{K}_0$ and $\mathbf{K}_1$, and we assume that $\mathbf{K}_1$ is small and can be treated perturbatively.

The characteristic relaxation times of the system, $\tau_n = \lambda_n^{-1}$, where $(-\lambda_n)$ are the eigenvalues of the Fokker–Planck operator

$$O = \nabla \cdot (-\mathbf{K} + \theta \nabla) \tag{5}$$

To each λ_n is associated a right-hand eigenfunction, Φ_n, and a left-hand one, Ψ_n:

$$O\Phi_n = -\lambda_n \Phi_n, \qquad O^+ \Psi_n = -\lambda_n \Psi_n \tag{6}$$

The zeroth-order Fokker–Planck operator $O^{(0)}$, corresponding to the potential force $\mathbf{K}_0$, becomes Hermitian under the transformation

$$\tilde{O}^{(0)} = \exp(U/2\theta) O^{(0)} \exp(-U/2\theta)$$

This entails that its normalized eigenfunctions have the form[1]

$$\Phi_n^{(0)} = \varphi_n \varphi_0, \qquad \Psi_n^{(0)} = \varphi_n / \varphi_0 \tag{7}$$

where the φ_n's are the normalized eigenfunctions of $\tilde{O}^{(0)}$:

$$\tilde{O}^{(0)} \varphi_n = \theta\left[\nabla^2 - (\nabla^2 \varphi_0)/\varphi_0\right]\varphi_n = -\lambda_n^{(0)} \varphi_n \tag{8}$$

$$\varphi_0 = (P_{st})^{1/2} = N^{1/2} \exp(-U/2\theta) \tag{9}$$

The corresponding eigenvalues, $-\lambda_n^{(0)}$, (common to $O^{(0)}$ and $\tilde{O}^{(0)}$) are real and positive, except for $\lambda_0^{(0)} = 0$, which is associated with the stationary state.

The Φ_n and Ψ_n sets are biorthogonal, and we assume that these functions form a complete set, i.e.,

$$\sum_n \Psi_n^*(\mathbf{r}) \Phi_n(\mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0) \tag{10}$$

Then, Eq. (6) can be developed into a standard perturbation expansion, and one obtains, for the first nonzero eigenvalue, in which we are interested here,

$$\lambda_1 = 1/\tau_K = \lambda_1^{(0)} + \lambda_1^{(1)} + \lambda_1^{(2)} + \cdots \tag{11}$$

where

$$\lambda_1^{(1)} = -\langle\Psi_1^{(0)}|O^{(1)}|\Phi_1^{(0)}\rangle \tag{12}$$

$$\lambda_1^{(2)} = \sum_{m\neq 1} \frac{\langle\Psi_1^{(0)}|O^{(1)}|\Phi_m^{(0)}\rangle\langle\Psi_m^{(0)}|O^{(1)}|\Phi_1^{(0)}\rangle}{\lambda_1^{(0)} - \lambda_m^{(0)}} \tag{13}$$

and the perturbation operator is given by

$$O^{(1)} = -\nabla\cdot\mathbf{K}_1 \tag{14}$$

Consider a matrix element of $O^{(1)}$. It can be written

$$(O^{(1)})_{mn} = \langle\Psi_m^{(0)}|O^{(1)}|\Phi_n^{(0)}\rangle = -\int d\mathbf{r}\, \frac{\varphi_m}{\varphi_0}\nabla\cdot(\mathbf{K}_1\varphi_n\varphi_0) \tag{15}$$

Developing the divergence in Eq. (15), and making use of relation (4b), we find

$$(O^{(1)})_{mn} = \int d\mathbf{r}\, \frac{\varphi_m}{\varphi_0}\mathbf{K}_1\cdot[\varphi_n\nabla\varphi_0 - \varphi_0\nabla\varphi_n] \tag{16a}$$

$$= -\int d\mathbf{r}\, \varphi_m\varphi_0\mathbf{K}_1\cdot\nabla(\varphi_n/\varphi_0) \tag{16b}$$

i.e., integrating by parts

$$(O^{(1)})_{mn} = -(O^{(1)})_{nm} \tag{17}$$

From this relation, it follows that $\lambda_1^{(1)} = 0$. Moreover, one checks immediately from equations (15) or (16b) that $(O^{(1)})_{m0} = (O^{(1)})_{0m} = 0$ (this expresses the fact that $\Phi_0 = \varphi_0^2$ is the *exact* lowest eigenfunction on both O and $O^{(0)}$). So, up to terms of third order in $\mathbf{K}_1$:

$$\lambda_1 - \lambda_1^{(0)} = \sum_{m\geqslant 2} \frac{|\langle\Psi_1^{(0)}|O^{(1)}|\Phi_m^{(0)}\rangle|^2}{\lambda_m^{(0)} - \lambda_1^{(0)}} \tag{18}$$

It can be noticed that expression (18) satisfies Risken's theorem: indeed, for $m \geqslant 2$, $\lambda_m^{(0)} - \lambda_1^{(0)} > 0$, so that $\lambda_1 > \lambda_1^{(0)}$, whatever $\mathbf{K}_1$ satisfying condition (4b).

In order to calculate explicitly expression (18), we need to know the $\lambda_n^{(0)}$'s and φ_n's. As discussed at length in Ref. 5, these can be calculated analytically when a generalized (multidimensional) WKB approximation is valid. This implies the following conditions, which we assume to be satisfied here:

(1) $\Delta U = U(0) - \max(U(a), U(b)) \gg \theta$ (small fluctuations)

(2) The two minima of U are connected by a well-defined "smooth" most probable path (MPEP): that is, the valley of U along the MPEP must have a slowly varying width, and a weak curvature and twist.

For simplicity, we will restrict ourselves to the case of a system described by only two stochastic variables (x, y), with a straight MPEP along the x axis, i.e., $\mathbf{a} = (a, 0)$, $\mathbf{b} = (b, 0)$. These last restrictions can be relaxed easily and do not affect the qualitative features of the result.

We want to calculate

$$(O^{(1)})_{m1} = \int d\mathbf{r}\, \frac{\varphi_m}{\varphi_0} \mathbf{K}_1 \cdot [\varphi_1 \nabla \varphi_0 - \varphi_0 \nabla \varphi_1] \tag{19}$$

It is known from previous work[5] that φ_0 and φ_1 have (in the only region where they have a nonnegligible amplitude, i.e., along the MPEP) the form

$$\varphi_{0,1}(\mathbf{r}) = f_{0,1}(x) \alpha_0(x) D_0\left(y \left[\frac{w(x)}{\theta} \right]^{1/2} \right) \tag{20}$$

where D_0 is the zeroth-order Weber function $[D_0(z) = \exp(-z^2/4)]$, $w(x)$ is the local transverse curvature of U along the MPEP:

$$U(x, y) \cong u(x) + \tfrac{1}{2} y^2 w(x) \tag{21}$$

and

$$\alpha_0(x) = \left[w(x)/2\pi\theta \right]^{1/4} \tag{22}$$

is chosen to normalize the y part of $\varphi_{0,1}$.

From Eq. (20), it results immediately that

$$\varphi_1 \frac{\partial \varphi_0}{\partial y} - \varphi_0 \frac{\partial \varphi_1}{\partial y} = 0$$

On the other hand, multiplying Eq. (8) by φ_0, we obtain

$$\nabla \cdot (\varphi_1 \nabla \varphi_0 - \varphi_0 \nabla \varphi_1) = \frac{\lambda_1^{(0)}}{\theta} \varphi_0 \varphi_1 \tag{23}$$

from which

$$\int_{-\infty}^{\infty} dy \left(\varphi_1 \frac{\partial \varphi_0}{\partial x} - \varphi_0 \frac{\partial \varphi_1}{\partial x} \right)_{x = x_0} = \frac{\lambda_1^{(0)}}{\theta} \int_{x < x_0} dx\, dy\, \varphi_0 \varphi_1 \tag{24}$$

So, the matrix element [Eq. (19)] reduces to

$$(O^{(1)})_{m1} = \frac{\lambda_1^{(0)}}{\theta} \int d\mathbf{r}\, \frac{\varphi_m(\mathbf{r})}{\varphi_0(\mathbf{r})} K_{1x}(\mathbf{r}) \alpha_0^2(x) \left[D_0\left(y \left[\frac{w(x)}{\theta} \right]^{1/2} \right) \right]^2 F(x) \tag{25a}$$

where

$$F(x) = \int_{-\infty}^{x} dx'\, f_0(x') f_1(x') \tag{25b}$$

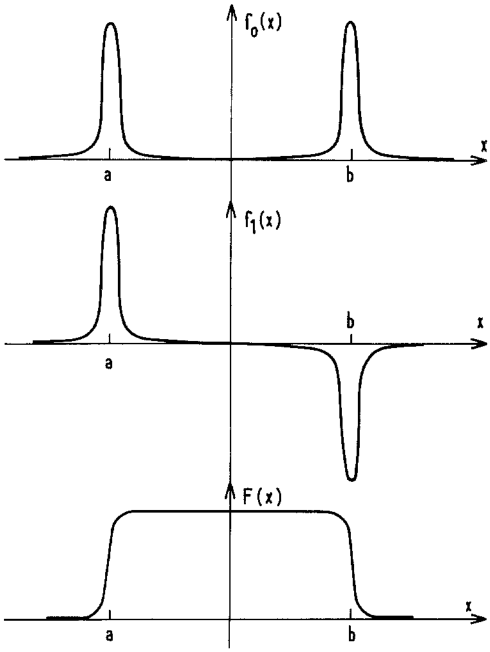

Fig. 1. Qualitative shape of $f_0(x)$, $f_1(x)$ and $F(x)$ for a system with a symmetric stationary distribution (symmetric U).

The functions f_0 and f_1 have narrow peaks of widths $\sim(\theta/w)^{1/2}$ in the vicinity of the minima of U $(x = a, b)$, and are exponentially smaller everywhere else[5,6] (see Fig. 1). Therefore, $F(x)$ is negligible for $x < a$ and $x > b$, and has a quasiconstant plateau between these two points.

Developing Eq. (9) around the minima of U, one gets for φ_0 the following approximate expression:

$$f_0(x) \cong N^{1/2}\left\{\exp\left[\frac{-U(\mathbf{a})}{2\theta}\right]\left(\frac{2\pi\theta}{w_a}\right)^{1/4} D_0\left((x-a)\left[\frac{u_a''}{\theta}\right]^{1/2}\right)\right.$$
$$\left. + \exp\left[\frac{-U(\mathbf{b})}{2\theta}\right]\left(\frac{2\pi\theta}{w_b}\right)^{1/4} D_0\left((x-b)\left[\frac{u_b''}{\theta}\right]^{1/2}\right)\right\} \tag{26}$$

with

$$u''_{a,b} = \left.\frac{\partial^2 U}{\partial x^2}\right|_{\mathbf{r}=\mathbf{a},\mathbf{b}}, \qquad w_{a,b} = \left.\frac{\partial^2 U}{\partial y^2}\right|_{\mathbf{r}=\mathbf{a},\mathbf{b}}$$

Analogously[3,4]:

$$f_1(x) = LD_0\left((x-a)\left[\frac{u_a''}{\theta}\right]^{1/2}\right) + MD_0\left((x-b)\left[\frac{u_b''}{\theta}\right]^{1/2}\right) \tag{27}$$

f_0 and f_1 must be normalized and orthogonal, which determines N, L, M. One then finds, for the plateau value C of $F(x)$,

$$\begin{aligned} C &\cong \int_{-\infty}^{0} f_0(x) f_1(x)\,dx \\ &= \exp\left[-\frac{U(\mathbf{a})+U(\mathbf{b})}{2\theta}\right] \\ &\quad \times \left\{(w_a u_a'')^{1/2}\exp\left[-\frac{U(\mathbf{b})}{\theta}\right] + (w_b u_b'')^{1/2}\exp\left[-\frac{U(\mathbf{a})}{\theta}\right]\right\}^{-1/2} \\ &\quad \times \left\{\frac{1}{(w_a u_a'')^{1/2}}\exp\left[-\frac{U(\mathbf{a})}{\theta}\right] + \frac{1}{(w_b u_b'')^{1/2}}\exp\left[-\frac{U(\mathbf{b})}{\theta}\right]\right\}^{-1/2} \end{aligned} \tag{28}$$

The eigenfunctions φ_m $(m \geqslant 2)$ which appear in Eq. (25a) separate into three classes[4]: the first one, $\{\varphi_{m_a}\}$, corresponds to states localized, along x, in a region of order $(\theta/u_a'')^{1/2}$ around point $\mathbf{a}$, the second one, $\{\varphi_{m_b}\}$, is correspondingly restricted to the (b) region, while the third class, $\{\varphi_{m_0}\}$, is localized in a region $\Delta x \sim (\theta/|u_0''|)^{1/2}$ around the saddle point $\mathbf{r}=0$ of U.

On the other hand, $[\varphi_o(r)]^{-1} \propto \exp(U/2\theta)$ is strongly peaked, along the MPEP, in the $x \cong 0$ region, where

$$[f_0(x)]^{-1} \cong N^{-1/2}\left(\frac{w_0}{2\pi\theta}\right)^{1/4} \exp\left[\frac{U(\mathbf{o})}{2\theta}\right] D_0\left(x\left[\frac{|u_0''|}{\theta}\right]^{1/2}\right) \tag{29}$$

and is very small around $\mathbf{a}$ and $\mathbf{b}$, so that the contribution of $\{\varphi_{m_a}\}$, $\{\varphi_{m_b}\}$ to $\lambda_1^{(2)}$ is exponentially small—and thus negligible—with respect to that of the φ_{m_0}'s, which, for $\mathbf{r} \cong 0$, have the form[5]

$$\begin{aligned} \varphi_{m_0}(\mathbf{r}) &\equiv \varphi_{n,p}^{(0)}(\mathbf{r}) \\ &\cong \left(\frac{|u_0''| w_0}{4\pi^2\theta^2}\right)^{1/4} (p!\,n!)^{-1/2} D_n\left(y\left[\frac{w_0}{\theta}\right]^{1/2}\right) D_p\left(x\left[\frac{|u''_0|}{\theta}\right]^{1/2}\right) \end{aligned} \tag{30}$$

This expression for the $\{\varphi_{m_0}\}$'s is obtained from Eq. (8) by linearizing the potential force $\mathbf{K}_0$ around $\mathbf{r}=0$. In order to be consistent with this

[4] Strictly speaking, this is true only for eigenvalues $\lambda_n^{(0)} \ll \theta^{-1}(\Delta U/a)^2$, which give the only important contribution to the sum (18).

approximation, we must also linearize $\mathbf{K}_{1x}$ in the same region. With

$$K_{0x} \cong |u_0''|x$$
$$K_{0y} \cong -w_0 y \tag{31}$$

we obtain, using condition (4b)

$$K_{1x} \cong \gamma y$$
$$K_{1y} \cong \gamma \frac{|u_0''|}{w_0} x \tag{32}$$

where γ is a (small) free parameter. So

$$(O^{(1)})_{m_{01}1} \cong \frac{\lambda_1^{(0)} C\gamma}{\theta} \exp\left[\frac{U(\mathbf{o})}{2\theta}\right] N^{-1/2}$$
$$\times \int dx \left(\frac{|u_0''| w_0}{4\pi^2\theta^2}\right)^{1/4} \frac{1}{(p!)^{1/2}} D_0\left(x\left[\frac{|u_0''|}{\theta}\right]^{1/2}\right) D_p\left(x\left[\frac{|u_0''|}{\theta}\right]^{1/2}\right)$$
$$\times \int dy \left(\frac{w_0}{2\pi\theta}\right)^{1/2} \frac{y}{(n!)^{1/2}} D_0\left(y\left[\frac{w_0}{\theta}\right]^{1/2}\right) D_n\left(y\left[\frac{w_0}{\theta}\right]^{1/2}\right) \tag{33}$$

It is seen immediately, with the help of the properties of the D_n's, that the only nonzero matrix element corresponds to $p = 0$; $n = 1$.

Finally, using the expression for the eigenvalues of the potential system[(5)5]

$$\lambda_{0,1}^{(0)} = w_0 + |u_0''| \tag{34}$$

$$\lambda_1^{(0)} = \frac{1}{\tau_K^{(0)}} = \frac{1}{2\pi}\left(\frac{|u_0''|}{w_0}\right)^{1/2} \exp\left[-\frac{U(\mathbf{o})}{\theta}\right]$$
$$\times \left\{ (w_a u_a'')^{1/2} \exp\left[\frac{U(\mathbf{a})}{\theta}\right] + (w_b u_b'')^{1/2} \exp\left[\frac{U(\mathbf{b})}{\theta}\right] \right\} \tag{35}$$

we find:

$$\frac{\lambda_1 - \lambda_1^{(0)}}{\lambda_1^{(0)}} \cong \frac{\gamma^2}{w_0(|u_0''| + w_0)} \tag{36}$$

This can also be expressed in terms of the Kramers times, τ_K and τ_K^0 of the nonpotential and potential systems, as

$$\tau_K \cong \tau_K^{(0)}\left[1 - \frac{\gamma^2}{w_0(|u_0''| + w_0)}\right] \tag{37}$$

[5] Note that the definition of the λ's used here differs from that of Ref. 5 by a factor θ.

This defines a posteriori the range of validity of our perturbation approximation as

$$\gamma \ll \left[w_0(|u_0''| + w_0) \right]^{1/2}$$

that is, in the linear region around the saddle point, $|\mathbf{K}_1| \ll |\mathbf{K}_0|$.

Therefore, we have shown that, to second order in perturbation, the variation of Kramers time due to the nonpotential part of the drift depends only on the characteristics of the force $\mathbf{K}_1$ in the saddle-point region and is independent of the details of its variations in the rest of the $\mathbf{r}$ space.

An analysis of the next perturbation orders parallel to the above one shows that this qualitative result persists up to rather high order in γ.

This simple qualitative result appears quite natural physically; indeed, Kramers dynamics can be understood in two ways:

(1) In the one used here, it is viewed as describing relaxation towards the stationary state.

(2) One can also, following Kramers,[7] view it as describing the current induced by an imposed chemical potential difference between the two locally stable states of the system. In this picture [if for example well (**b**) is assumed empty] one can write

$$\frac{1}{\tau_K} = \frac{J_{a \to b}}{n_a} \tag{38}$$

[where n_a is the population of well (**a**)]. For $\Delta U \gg \theta$, $J_{a\to b}$ is small, so that n_a is practically determined by the stationary distribution, common to the ($\mathbf{K}_0$) and ($\mathbf{K}_0 + \mathbf{K}_1$) systems. So, the difference between τ_K and $\tau_K^{(0)}$ is only due to the difference between the $J_{a\to b}$ in the two systems, which is clearly determined by the properties of the saddle-point region.

One is left with the question of understanding what is the physical meaning of Risken's theorem, i.e., why does the introduction of a nonpotential part in the drift force always result in a decrease of τ_K? We believe that a hint about this question can be found in the following remark: the population n_a in Eq. (38) is determined by P_{st}, i.e., by U. Therefore, *for populations*, side (**a**) is defined by the principal axes of U at the saddle point. With the geometry we chose here for U, this "stationary (**a**) side" corresponds to $x < 0$.

On the other hand, in the absence of fluctuations, the dividing line for trajectories in the $\mathbf{r} \cong 0$ region is fixed by the principal axes of the linearized total force, $\mathbf{K}_0 + \mathbf{K}_1$. Using Eqs. (31) and (32), it is seen that the "dynamic dividing line" of the deterministic trajectories is, for $\gamma \neq 0$, at an angle β with the "population dividing line" ($\beta \simeq \gamma/(|u_0''| + w_0)$).

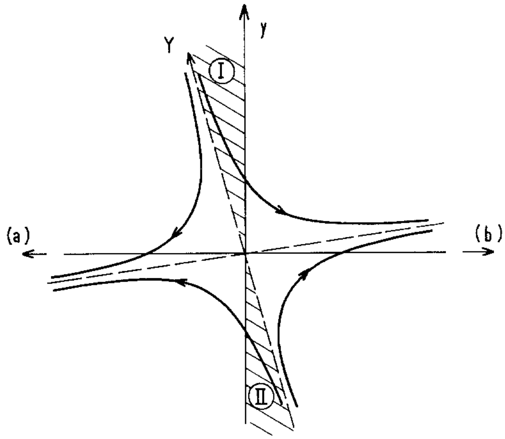

Fig. 2. Qualitative shape of the deterministic trajectories near the saddle-point 0. $0y$ is the stationary dividing line. $0Y$ is the dynamic dividing line.

In the potential system, where the two lines coincide, all the particles which cross the dividing line have to do so with the help of the fluctuations. In the nonpotential system, there always is a region of the stationary (**a**) side from which particles are driven across the stationary dividing line by the deterministic motion (Fig. 2).[6]

This cannot be compensated by the deterministic motion towards (**a**) issuing from region (II) of Fig. 2, since, due to the chemical potential difference, the population in this region is smaller than that of region (I).

This picture thus seems to account, at least qualitatively, for Risken's effect.

Finally, it is interesting to compare our result with what Gardiner's method[(3)] would predict in the same case. He characterizes the nonpotential part of the force by means of an antisymmetric tensor $\bar{\bar{A}}(x, y)$, related to our $\mathbf{K}_1$ by

$$\mathbf{K}_1 = \nabla \cdot \bar{\bar{A}} + \frac{1}{P_{st}} (\nabla P_{st}) \cdot \bar{\bar{A}} \tag{39}$$

and introduces a set of planes $\{S(\xi)\}$ chosen so that, in each S, P_{st} has a unique maximum, at position $\mathbf{u}(\xi)$. Writing the equation of these planes as

$$\mu(\xi)x + y = \xi \tag{40}$$

[6] Of course, in order to calculate τ_K explicitly on this basis, one should take into account the fact that, in the saddle-point region, the fluctuation effect is important and couples with the deterministic one.

the orientation $\mu(\xi)$ of $S(\xi)$ is fixed by the condition

$$\left[\bar{\bar{\theta}}^T + \bar{\bar{A}}^T(\mathbf{u}(\xi))\right] \cdot \hat{n}(\xi) = d(\xi)\frac{d\mathbf{u}(\xi)}{d\xi} \tag{41}$$

where $\hat{n}(\xi)$ is the unit vector normal to $S(\xi)$.

One easily calculates explicitly $\mathbf{u}(\xi)$ and $\mu(\xi)$ in the quadratic region $\mathbf{r} \cong 0$, and, following Gardiner,[3,8] we find that the expression thus obtained for the Kramers time in the small-γ limit is identical with our perturbation result [Eq. (37)].

This can be considered as a check of the validity of Gardiner's assumptions. Moreover, for $\mathbf{r} \cong 0$ ($\xi \cong 0$), one finds $\mu = \gamma/(|u_0''| + w_0) = \beta$ that is, the set of planes that he introduces precisely follow the principal axis of the force field which is transverse to the dynamic most probable path, and $S(\xi = 0)$ coincides with the "dynamic dividing line."

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